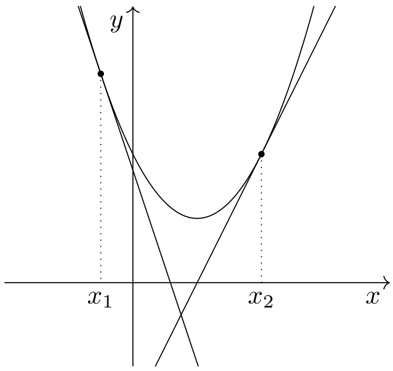
3301. A function f is defined over the reals, and has range $[a, b]$, where $0 < a < b$. Give the range of each of the following functions, over its broadest possible domain:

(a) $x \mapsto \mathrm{f}(x) + k$,
(b) $x \mapsto \big(\mathrm{f}(x)\big)^2$,
(c) $x \mapsto \dfrac{1}{\mathrm{f}(x)}$.

3302. On a parabola $y = \mathrm{f}(x)$, two tangents are drawn, at points with x coordinates p and q:



Prove that the x coordinate of the intersection of the tangent lines is the mean of p and q.

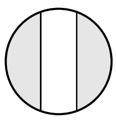
3303. Find the set of values of x for which both of the functions $\mathrm{f}(x) = 20 + x - x^2$ and $\mathrm{g}(x) = x^3 - 6x^2$ are decreasing.

3304. Prove that the number of ways of selecting a set of r objects from among n objects is given by

$$\frac{n!}{r!(n-r)!}.$$

3305. Show that $\displaystyle\int_{-1}^{1} \frac{3x^2 + 18x + 26}{x^3 + 9x^2 + 26x + 24}\, dx = \ln 10$.

3306. A circular icon is shaded in stripes of equal width.



Show that around 58% of the icon's area is shaded.

3307. One of the following statements is true; the others are not. Prove the one and disprove the others.

(a) $\sin^2 x + \sin x + 1 = 0$ has no real roots,
(b) $\sin^2 x + \cos x + 1 = 0$ has no real roots,
(c) $\sin^2 x + \tan x + 1 = 0$ has no real roots.

3308. Three coins are tossed. Then, if there are more heads than tails, all three are tossed again. This continues until there are more tails than heads. Find the probability that the end result is all tails.

3309. Show that the hyperbolae $(x-2)(y-2) = 1$ and $xy = 1$ are tangent to one another.

3310. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

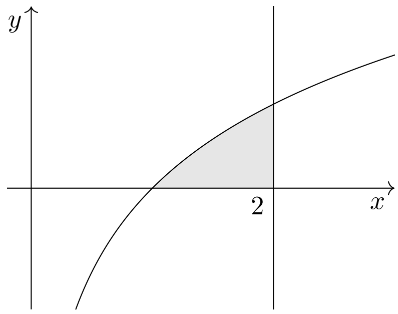
① $\mathrm{f}(x)$ has a factor of $(x-p)^2$,
② $\mathrm{f}(x)$ has a stationary point at $x = p$.

3311. A curve is given as $x^3 - y^8 = 1$.

(a) Find the equation of the tangent at $(1, 0)$.
(b) Show that $x \geq 1$.
(c) Hence, sketch the curve.

3312. Find the probability that two numbers chosen at random in the interval $[0, 1]$ differ by less than $\frac{1}{2}$.

3313. The graph shows a region R enclosed by the curve $y = \ln x$, the line $x = 2$, and the x axis:



Determine the exact area of R, giving your answer in the form $A = \ln a - b$.

3314. Two lines have equations

$$L_1 : y = 2x - 8,$$
$$L_2 : y = -2x.$$

Find, in the form $(x-a)(y-b) = 0$, the equation of the locus of points equidistant from L_1 and L_2.

3315. Show that, for small A in radians,

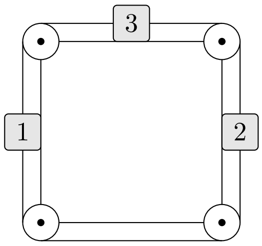
(a) $\cos\left(\frac{\pi}{2} + A\right) \approx -A$,
(b) $\tan\left(\frac{\pi}{4} + A\right) \approx 1 + \dfrac{2A}{1-A}$.

3316. Curve C has equation $y = x^{\frac{2}{3}} + x^{\frac{1}{3}}$.

(a) Find any x axis intercepts.
(b) Find and classify any stationary points.
(c) Show that C is tangent to the y axis at O.
(d) Verify that C is concave on $(-\infty, -1) \cup (0, \infty)$.
(e) Hence, sketch the curve.

3317. Distinct integers n_1 and n_2 are randomly chosen from the set $\{1, 2, ..., 10\}$. Find $\mathbb{P}(|n_1 - n_2| = 1)$.

3318. A pulley system consists of three masses attached to a loop of light, inextensible string, which is passed around four smooth pulleys at the corners of a smooth, vertical square. Masses are in kg.



(a) Explain why it is not possible to find tensions with the information given.

(b) Find the acceleration of the system.

3319. The following curve is stationary at $y = \pm 1$:

$$y = \frac{kx}{(x^2+3)^2}.$$

Determine all possible values of the constant k.

3320. Solve the equation $\sqrt{2x+1} + \sqrt{2x-2} = 3$.

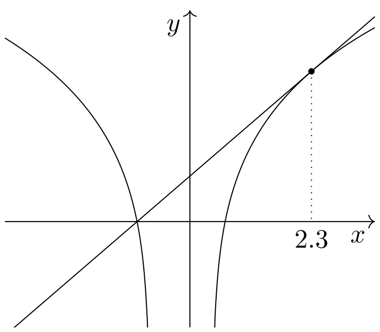
3321. Two circles are given by $x^2 + y^2 + 2y - 3 = 0$ and $x^2 - x + y^2 = 0$. Prove that the shortest distance between the circles is along the line $y = 2x - 1$.

3322. With the sine function taking inputs in radians, set S is defined as follows:

$$S = \left\{x \in \mathbb{Q} : \sin\left(x^2\right) > \tfrac{1}{2}\right\}.$$

Prove that S has infinitely many elements.

3323. A function is given as $\mathrm{f}(x) = \ln\left(3x^2 + x - 1\right)$. A tangent is drawn to $y = \mathrm{f}(x)$ at $x = 2.3$:



Show algebraically that, as in the diagram above, the tangent meets the x axis close to a root of f.

3324. The cubic function $\mathrm{f}(x) = x^3 + ax + b$ is invertible on the domain $\mathbb{R}$. Find all possible values of the constants a and b, answering in set notation.

3325. Find a simplified expression for $\displaystyle\int \frac{x^2-4x+1}{x^2-4x+4}\,dx$.

3326. Disprove this statement: "All polynomial curves have either reflective or rotational symmetry."

3327. A camera of mass 99 kg is mounted on a tripod, which is rigid and of negligible mass. The legs of the tripod are 100 cm long, and their feet stand at the three vertices of an equilateral triangle of side length $10\sqrt{3}$ cm.

(a) Find the exact angle between each of the legs and the vertical.

(b) Determine the exact thrust T in each leg, in the form $T = g\sqrt{a}$ N, where $a \in \mathbb{Z}$.

3328. Triangle T has an obtuse angle and sides of length $(a-1, a, a+1)$. Show that $a < 4$.

3329. The numbers 1 to 9 are placed at random in a 3×3 grid, as in the example shown:

5	8	2
6	7	3
4	9	1

Find the probability that, in a new placement, the squares containing 1 and 2 share an edge.

3330. For constants $p, q \in \mathbb{N}$, a region R of the (x, y) plane is defined by three simultaneous inequalities:

$$x^p y^q \le 1, \qquad x \ge 0, \qquad y \ge 0.$$

(a) Explain why the area of R is given by

$$A = \lim_{a \to 0} \int_a^{\frac{1}{a}} x^{-\frac{q}{p}}\,dx.$$

(b) Show that $A = \frac{q}{p+q} \lim_{a\to 0} \left(a^{\frac{p+q}{q}} - a^{-\frac{p+q}{q}}\right)$.

(c) Show that, whatever the values of $p, q \in \mathbb{N}$, region R has infinite area.

3331. Find the range of the function

$$\mathrm{g}(\theta) = \frac{34}{8\sin 2\theta + 15\cos 2\theta}.$$

3332. Prove that no polynomial function of even degree can be increasing everywhere.

3333. Solve for x in the inequality $(x-a) \ge (x-a)^2$. Give your answer in set notation, in terms of the constant a.

3334. It is given that $a, b, c, d > 0$ are in increasing GP. Prove that $a + d > b + c$.

3335. Three dice have been rolled, giving scores X, Y, Z. Show that knowing the fact "$XYZ = 6$" increases the probability that $X = 1$.

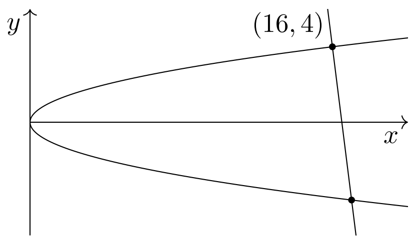
3336. Show that $\displaystyle\sum_{i=1}^{n} \ln\left(1 + \frac{1}{i}\right) \equiv \ln(n+1)$.

3337. Variables p, q, x, y are linked by

$$\begin{aligned} y &= p^2, \\ p &= q^2 + q, \\ q &= x - 1. \end{aligned}$$

Sketch y against x, marking axis intercepts.

3338. The curve $x = y^2$ is shown, together with a normal to the curve at $(16, 4)$.



Determine the y coordinate of the point at which the normal re-intersects the curve.

3339. This question concerns a proof of the formula for the volume of a cone.

A straight line in the (x, y) plane, passing through the origin and the point (h, r), is rotated around the x axis in (x, y, z) space. The following integral is set up:

$$V_{\text{cone}} = \int_0^h \pi \left(\tfrac{r}{h}x\right)^2 \, dx.$$

(a) Interpret the quantity $\frac{r}{h}x$.

(b) Give the physical meaning of the integrand.

(c) Prove that $V_{\text{cone}} = \frac{1}{3}\pi r^3 h$.

3340. A sample $\{x_i\}$ has standard deviation $s_x = 0.1$ and mean 3.1. Its greatest value is $x_{\max} = 4.0$ and its least value is $x_{\min} = 2.3$. Show that one of the quantities below can be guaranteed to be larger than the other:

① $\sum (x_i - \bar{x})^2$

② $\sum |x_i - \bar{x}|$.

3341. The quadratic graph $y = x^2 + x - 12$ is reflected in the line $x = a$, yielding $y = x^2 + bx$. Determine all possible values of a and b.

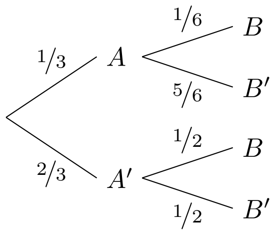
3342. A student integrates, by inspection, as follows:

$$\int \frac{4x}{x^2+1} \, dx = 4 \ln|x^2 + 1| + c.$$

Explain the error, and correct it.

3343. Express $\tan(\arcsin x)$ in a form not involving any trigonometric functions.

3344. Events A and B have probabilities as represented on the following tree diagram, conditioned on A.



Draw an equivalent tree diagram conditioned on B, calculating all of the branch probabilities.

3345. A function is given, for $2 \leq k \in \mathbb{N}$, by

$$\mathrm{g} : x \mapsto x^{2k} + x^{2k+1}.$$

(a) Show that the second derivative is

$$\mathrm{g}''(x) = 2kx^{2k-2}\big((2k-1) + (2k+1)x\big).$$

(b) Hence, show that $\mathrm{g}''(0) = 0$, but that $x = 0$ is not a point of inflection of $y = \mathrm{g}(x)$.

3346. It is given that $y = \mathrm{f}(x)$ has rotational symmetry around the point $(1, 0)$, and also that

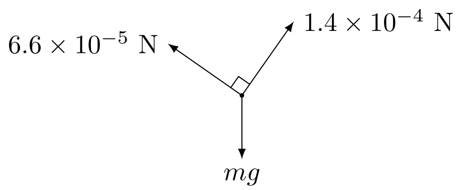
$$\int_0^1 \mathrm{f}(x) \, dx = k.$$

Find the following integrals, in terms of k:

(a) $\displaystyle\int_0^2 \mathrm{f}(x) \, dx$,

(b) $\displaystyle\int_2^1 \mathrm{f}(x) \, dx$.

3347. In a lab, a dust particle, mass $m = 13$ milligrams, is accelerating vertically under the action of a pair of perpendicular electromagnetic forces.



Determine the magnitude of the acceleration.

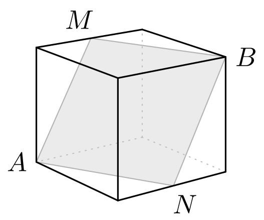
3348. Show that $xy^3 - 2x = 3y$ cannot be expressed as $y = \mathrm{f}(x)$, where f is a function.

3349. Two dice are rolled together. They have m sides and n sides respectively, where $m < n$. Find, in terms of m and n, the probability p that the score on the m-sided die is larger.

3350. True or false?

(a) No two primes differ by three,
(b) No two primes differ by five,
(c) No two primes differ by seven.

3351. The diagram shows a unit cube and a quadrilateral formed of two vertices and two midpoints.



(a) Describe quadrilateral $AMBN$.
(b) Find the area of $AMBN$.

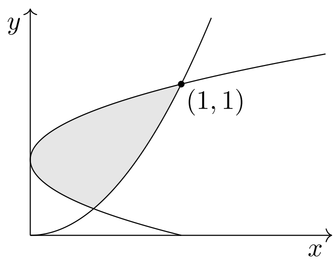
3352. Unit circles with centres $(0, 0)$ and $(0, k)$ intersect each other at right-angles. Find all possible values of the constant k.

3353. In mechanics, a *couple* is defined to be a pair of equal and opposite forces whose line of action is not the same. Prove that the turning effect of a couple is the same around any pivot point.

3354. Find $\displaystyle\int 8x \cos x \sin x \, dx$.

3355. Region R of (x, y) points is defined by $R = A \cap B$, where A is the set of points satisfying $x > 1 - |y|$, and B is the set of points satisfying $x^2 + y^2 \leq 1$. Show that the area of region R is $\frac{\pi}{2} - 1$.

3356. The parabolae shown below have equations $y = x^2$ and $x = (2y - 1)^2$. They intersect at $(1, 1)$.



Find the y coordinate of other intersection point, and hence show that the area of the shaded region is 0.405, to 3sf.

3357. A function f has a definite integral, between $x = a$ and $x = b$, given by

$$\int_a^b \mathrm{f}(x) \, dx = I.$$

Evaluate, in terms of a, b and I:

(a) $\displaystyle\int_b^a \mathrm{f}(x) - 1 \, dx$,
(b) $\displaystyle\int_{\frac{b}{3}}^{\frac{a}{3}} \mathrm{f}(3x) \, dx$.

3358. By expanding $(1 - x)^{\frac{1}{2}}$ and substituting $x = 0.02$, show that $\sqrt{2} \approx \frac{99}{70}$.

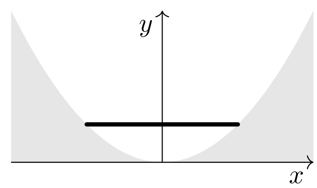
3359. An iteration I is defined, for some $k \in \mathbb{N}$, by

$$x_{n+1} = \sum_{r=1}^{2k+1} x_n^r.$$

Show that I has at least two fixed points.

3360. Solve $\operatorname{cosec} 2x + \sec 2x = 0$, for $x \in [0, 2\pi)$.

3361. A uniform rod of weight W and length 2 units is resting in equilibrium on the smooth inner surface of a parabolic bowl. In cross-section, the bowl's shape is modelled as $y = x^2$, as shown:



(a) Show that $\sin\left(\arctan \frac{1}{2}\right) = \frac{1}{\sqrt{5}}$.
(b) Find, in exact terms of W, the contact force exerted on the bowl by each end of the rod.

3362. Prove that, if two cubics of the form $y = \mathrm{f}(x)$ are tangent to each other and cross at their point of tangency, then they cannot intersect elsewhere.

3363. Write $x^4 + 2x^3 + x^2 - 1$ in terms of $z = x^2 + x + 1$.

3364. In a circle of radius r, a chord of variable length c subtends a variable angle θ in radians at the centre of the circle. Prove that

$$\frac{dc}{dt} = \frac{r^2}{c} \sin\theta \frac{d\theta}{dt}.$$

3365. On (x, y) axes, shade the region(s) that satisfy the inequality $|x - y|(x + y) \geq 0$.

3366. Find a quintic approximation, valid for small x, to

$$\mathrm{f}(x) = \frac{x}{1 - x^2}.$$

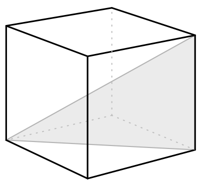
3367. A projectile is launched from ground level at speed u, at an angle randomly chosen on $[0, 90°]$. Find the probability that the range d of the particle will satisfy

$$d \geq \frac{u^2}{2g}.$$

3368. Find $\displaystyle\int e^{\sin t} \cos t \, dt$.

3369. The curve $y = x^2$ has a normal $y = mx + c$ drawn to it at a point with $x \neq 0$. Prove that $c > 1$.

3370. The diagram below shows a cube. A triangle has been formed by choosing three distinct vertices:



Prove that there are three non-congruent triangles which can be formed in this way.

3371. A pub landlord is setting up a weekly lottery for charity. Three ales are to be selected at random from the twenty on the menu. Entrants will pay £1, and guess the three ales. Guessing two ales correctly wins a second prize of £5, and guessing all three correctly wins a first prize of £n, where n is to be decided. The landlord wants the expected total amount paid out for first and second prizes to be similar.

(a) Find $\mathbb{P}$(matching two ales).
(b) Hence, determine the average payout, per ticket bought, on second prizes.
(c) Find $\mathbb{P}$(matching three ales).
(d) Hence, suggest an appropriate top prize.

3372. Prove that no $n+1$ distinct points on a polynomial graph $y = \mathrm{f}(x)$ of degree $n \geq 2$ are collinear.

3373. A taut, light string is passed over a peg, which is modelled as a smooth cylinder. The string has an external force applied at each of its ends. Explain why the magnitudes of the applied forces must be equal.

3374. State, with a reason, whether the following hold, regarding functions f and g:

(a) "A fixed point of both f and g is a fixed point of the composition fg."
(b) "A root of both f and g is a root of fg."
(c) "A stationary point point of both f and g is a stationary point of fg."

3375. By considering an ellipse as a transformed circle, find the exact area of the region of the (x, y) plane defined by the inequality $2x^2 + 5y^2 \leq 10$.

3376. Prove that $\dfrac{2\tan^2 x}{\tan^2 x + 1} \equiv 1 - \cos 2x$.

3377. Take $g = 10$ in this question.

A particle is projected over horizontal ground, from an origin $O : (0, 0, 0)$, with initial velocity $\mathbf{u} = 20\mathbf{i} + 20\mathbf{j} + 10\mathbf{k}$ ms^{-1}.

(a) Find the coordinates of the highest point.
(b) Show that the particle lands at 30 ms^{-1}.
(c) Find the angle below the horizontal at which the particle is moving when it lands.

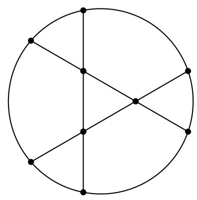
3378. A curve is given as $y^3 = x - x^2$.

(a) Show that $3y^2 = (1 - 2x)\frac{dx}{dy}$.
(b) Hence, show that, at O, the tangent is parallel to the y axis.
(c) By differentiating the result in (a) with respect to y, show that O is a point of inflection.
(d) Hence, sketch the curve in the vicinity of O.

3379. Show that, for $a < b \in \mathbb{N}$,

$$2^a + 2^{a+1} + ... + 2^b \equiv 2^{b+1} - 2^a.$$

3380. A circle has three symmetrical chords drawn to it, such that the length of each of the nine straight line segments so formed is 1.



Find the area of the circle.

3381. Prove that no polynomial graph $y = \mathrm{f}(x)$ can be both concave and increasing on $(0, \infty)$, and also stationary at the origin.

3382. By considering the multiplicity of its roots, sketch $y = \cos^3 x$.

3383. Four fair dice are rolled. Determine which of the following events has the greater probability:

① two pairs,
② three of a kind.

3384. Find all fixed points of the iteration

$$x_{n+1} = \frac{1}{x_n - 1} + \frac{1}{(x_n - 1)^2}.$$

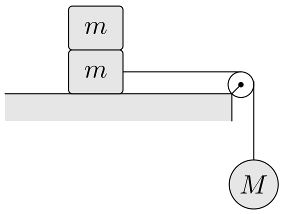
3385. The equation $ax - 8\sqrt{x} + 4 = 0$, where $a \in \mathbb{R}$ is a constant, has exactly one real root. Find all possible values of a, by considering the cases

① $a > 0$,

② $a = 0$,

③ $a < 0$.

3386. Two blocks are stacked on a table. The lower block is connected by a light string, which is passed over a smooth, light, fixed pulley, to a sphere which hangs freely. Both contacts with the lower block are rough, with coefficient of friction μ.



(a) State a further assumption needed to model the objects connected by the string as having the same acceleration.

(b) Show that the upper block will not have the same acceleration as those connected by the string if

$$M \geq \frac{2\mu m}{1-\mu}.$$

(c) Interpret this formula in the case $\mu \geq 1$.

3387. Two polynomial curves, which do not intersect, are drawn in an (x, y) plane. Prove by contradiction that the shortest path between the curves must be perpendicular to both.

3388. A differential equation is given as

$$\frac{dy}{dx} - \frac{3y}{x} = \frac{e^x}{x^3}.$$

A solution curve $y = \dfrac{e^x - e}{x^3}$ is suggested.

(a) Find $\dfrac{dy}{dx}$ for the proposed curve.

(b) By substituting, determine whether the curve does satisfy the differential equation.

3389. The first three terms of an arithmetic progression are k, k^3, k^4. Find all possible values of k.

3390. Show that no tangent to $y = e^{-x^2}$ goes through O.

3391. Two monkeys of mass m are holding onto opposite ends of a light rope, which is passed over a rough pulley. One then begins to climb the rope. Explain what happens, with reference to Newton's laws.

3392. Find $\displaystyle\int \sec x(\tan x + \sec x)\, dx$.

3393. Solve ${}^n\mathrm{C}_3 - 2 \cdot {}^n\mathrm{C}_2 = 12$, for $n \in \mathbb{N}$.

3394. Curve C is given by $y = \dfrac{1 + \cos 2x}{2 + \sin 2x}$.

(a) Solve to find x intercepts.

(b) Show that $y \geq 0$ everywhere.

(c) Find the gradient of the curve C at x values halfway between the x intercepts.

(d) Hence, sketch C. You don't need to find the coordinates of the local maxima.

3395. Four couples sit down at random at a round table. Find the probability that everyone ends up sitting opposite their partner.

3396. Take $g = 10$ in this question.

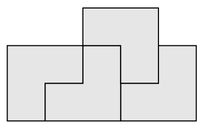
A particle is projected from 2 m above horizontal ground, so as to hit a target at ground level, 2 m away horizontally. Its initial velocity is 5 ms^{-1} at a positive angle θ above the horizontal.

(a) Show that $2\sec^2\theta - 5\tan\theta - 5 = 0$.

(b) Hence, find the exact angle of projection.

3397. The decimal expansion of x terminates after $m > 0$ digits. Prove that the decimal expansion of x^n, for $n \in \mathbb{N}$, terminates after mn digits.

3398. A children's game consists of four tiles, as below.



Prove that the tiles cannot be arranged to form a solid square.

3399. Two graphs are defined, for $t \in \mathbb{R}$, by the equations

$$\mathbf{r} = \begin{pmatrix} 3\cos t \\ 3\sin t \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 5 - t \\ 1 - \frac{1}{2}t \end{pmatrix}.$$

Show that the graphs intersect at $(3, 0)$, and find the other point of intersection.

3400. Verify, by differentiating, that

$$\int \cos(\ln x)\, dx = \tfrac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + c.$$

——— End of 34th Hundred ———